APPLICATION OF WALSH FUNCTIONS TO CONSTRUCT AN EXPLICIT PROJECTION ALGORITHM FOR IDENTIFICATION OF DISTRIBUTED SYSTEMS

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High interference-immunity and resolution are obtained for an explicit projection identification algorithm that is realized easily on an electronic computer.

Many physical phenomena and technical objects are modeled by partial differential equations. These models describe the evolution of systems in time and their distributivity in space. It is customary to call such systems systems with distributed parameters (SDP). Following [1], SDP models can be considered as "input—output" type models where a perturbing system of functions (heat source distribution, heating rods or a force, a perturbing fluctuation) with initial and boundary conditions is the input and the output function of a system (temperature distribution in a heated rod or displacement of a vibrating body) is the output. Therefore, the model connects the input to the output function by means of a system operation which is a partial differential operator. The output function depends on both the input function and on its internal parameters that enter into the system operator in the form of factors (for instance, the heat conduction coefficient or the rate of vibration propagation).

Ordinarily not all the internal parameters are given in a real system. Consequently, unknown parmeters must be estimated for its simulation, diagnostics, optimization, or control, i.e., parametric identification of the system must be performed. A given system input and the output observable with errors are usually the initial data for the SDP identification problem. Starting from physical laws and operating conditions, the structure of a mathematical model of the SDP is postulated here to the accuracy of the unknown internal parameters.

Existing methods of parametric SDP identification can be separated into explicit and implicit [2]. Explicit methods are associated with minimization of the quality criterion from the residual of the equation (the residual in the input) dependent explicitly on the parameters being estimated. When using implicit identification methods the quality criterion is constructed according to the output residual which depends implicitly on the unknown parameters. Explicitly SDP parametric identification algorithms are easily realizable on a minicomputer; however, they are too unstable relative to small deviations from the initial data. Implicit algorithms possess high interference-immunity but their numerical realization is sufficiently tedious because of the need for a multiple solution of partial differential equation [3]. In this connection there is a need to synthesize new or to perfect known explicit SDP identification algorithms that possess higher interference-immunity than the existing ones.

It turns out that explicit algorithms of projection type satisfy these requirements. The difference between the proposed projection algorithm and those known is that in order to minimize the residual in the input it is projected on a subspace of piecewise-constant functions rather than on an arbitrary finite-dimensional projection subspace. It is proved in [2] that such a subspace filters out random errors present in the observable output. However, selection is possible even among subspaces generated by piecewise-constant functions.

The purpose of the paper is the selection and investigation of a projection subspace generated by a system of piecewise-constant Walsh functions that possesses the highest filtering properties and thereby increasing interference-immunity of the identification algorithm.

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FORMULATION OF THE PROBLEM

Considered are SDP simulatable by nonlinear second-order partial differential equations

$$\sum_{n=1}^{N}\Theta_n\varphi_n[D_\nu u(x,\ t)]=f(x,\ t), \tag{1}$$

where $x\in\Delta=[x_0,\ x_\Delta]$; $t\in\Lambda=[t_0,\ t_\Lambda]$; $D_\nu\in\left\{\frac{\partial^2}{\partial t^2},\ \frac{\partial^2}{\partial x\partial t},\ \frac{\partial^2}{\partial x^2},\ \frac{\partial}{\partial t},\ \frac{\partial}{\partial x},\ I\right\}$ $(\nu=\overline{1,\ 6})$ are partial differential operators that describe the system behavior, where I is the identity operator. Initial and boundary conditions are appended to the equation for its unique resolution relative to u(x, t).

It is assumed that the observable output function u(x, t) is measured at discrete points $\{x_m\}_1^M\in\Delta$, $\{t_k\}_1^K\in\Lambda$ and distorted by a certain additive ergodic random function $\varepsilon(x,\ t)$ that has zero mean and a covariational function absolutely integrable in the domain $\Delta\times\Lambda$, i.e.,

$$u'_{mk}=u'(x_m,\ t_k)=u^*(x_m,\ t_k)+\xi_{mk},\quad E\{\xi_{mk}\}=0,\quad \xi_{mk}=\xi(x_m,\ t_k), \tag{2}$$

where $u^*(x_m,\ t_k)$ is an exact, but unknown, output function, $E\{\cdot\}$ is the symbol for mathematical expectation. It is also assumed that values of the output function are measured on the boundaries of the interval Δ. Let a certain continuous function $\tilde u(x,\ t)$ that interpolates $\{u_{mk}'\}$ with negligible systematic interpolation error correspond to measurable values of $\{u_{mk}'\}$.

The problem is to estimate the unknown components $\{\Theta_n\}$ of the vector Θ by means of the measurements (2). We shall later assume that all N components of the vector Θ must be estimates.

CONSTRUCTION OF THE EXPLICITY PROJECTION IDENTIFICATION ALGORITHM

According to [2], the algorithm consists of two stages. In the first the system residual in the input is determined

$$l(x,\ t,\ \Theta)=\tilde f(x,\ t,\ \Theta)-f(x,\ t), \tag{3}$$

where $\tilde f(x,t,\Theta)=\sum_{n=1}^{N}\Theta_n\varphi_n[D_\nu\tilde u\ (x,t)]$ is the input function of the system model which is calculated to the accuracy of unknown coefficients of Θ by application of the operators D_ν from (1) to the function $\tilde u(x,\ t)$. It is assumed that $\ell(x,\ t,\ \Theta)$ is a square summable function in the domain $\Delta\times\Lambda$, i.e., an element of the Hilbert space $L_2(\Delta\times\Lambda)$.

As is known [4], the problem of numerical differentiation is an inverse incorrect problem. Consequently, to determine the residual of $\ell(x,\ t,\ \Theta)$ relative to the unknown Θ it is necessary to use a regularizing algorithm of numerical differentiation. Up to now sufficiently many such algorithms exist based particularly on spline-smoothing of experimental data [5, 6]. The proposed algorithm utilizes the numerical differentiation method from [7], which possesses high interference-immunity when there are 30 or more measurement points in 2-3 intervals of monotoneity of the differentiable function. Moreover, the method possesses the advantage that as the quantity of measurement points increases the error of the derivatives approximates the maximally achievable error at approximately the rate $N^{-1/2}$, where N is the number of measurement points. The maximally achievable error is understood to be the error in approximating the derivatives by a given finite set of linearly independent functions.

Therefore, the derivatives $D_\nu\tilde u(x,\ t)$, as well as the functions $\varphi_n[D_\nu\tilde u(x,\ t)]$, are calculated in the first stage of the algorithm.

The problem

$$\hat\Theta=\min_\Theta\|l(x,\ t,\Theta)\|^2=\min_\Theta\int_\Delta\int_\Lambda[l(x,\ t,\ \Theta)]^2\,dxdt \tag{4}$$

is solved in the second stage of the algorithm to obtain an estimate of the vector Θ, where $\|\cdot\|$ denotes the norm in the Hilbert space $L_2(\Delta \times \Lambda)$. As is seen, the residual (3) and the minimizable criterion (4) depend explicitly on the parameters being estimated; consequently, according to [8] this algorithm belongs to the class of explicit identification algorithms. We solve the problem (4) by a projection method [9].

We first determine the N-dimensional projection space F_N which is a subspace of the space $L_2(\Delta \times \Lambda)$ defined above. Let the system $\{W_m(x, t)\}_1^N$ be a basis in F_N. To minimize the residual $\ell(x, t, \Theta)$ we project the model input function $\tilde{f}(x, t, \Theta)$ on F_N by using the orthogonal projection operator P_N, defined in the Hilbert space $L_2(\Delta \times \Lambda)$, i.e., P_N: $L_2(\Delta \times \Lambda) \to F_N$. Then the residual $\ell(x, t, \Theta)$ is orthogonal to the subspace F_N and thereby to each function $W_m(x, t)$, $m(\overline{1, N})$, i.e., the scalar products

$$\langle l(x, t, \Theta), W_m(x, t) \rangle = \int_\Delta \int_\Lambda l(x, t, \Theta) W_m(x, t)\, dx dt, \quad m = \overline{1, N} \tag{5}$$

in $L_2(\Delta \times \Lambda)$ equal zero. Taking account of (3), we obtain a system of linear algebraic equations in the desired vector Θ:

$$\sum_{n=1}^{N} \Theta_n \langle \varphi_n [D_\nu \tilde{u}(x, t)], W_m(x, t) \rangle = \langle f(x, t), W_m(x, t) \rangle, \ m = \overline{1, N}. \tag{6}$$

If $A = \{a_{nm}\} = \{\langle \varphi_n [D_\nu \tilde{u}(x, t)], W_m(x, t) \rangle\}$ denotes the matrix of the system and $b = \{b_m\} = \langle f(x, t), W_m(x, t) \rangle$ the vector of the right side, then the estimate of the vector Θ is obtained in the form

$$\hat{\Theta} = A^{-1} b; \tag{7}$$

it minimizes the functional (4) on the subspace F_N. As is seen from (6) and (7), the matrix A of the obtained system is perturbed since the quantities $D_\nu \tilde{u}(x, t)$ are calculated with errors.

It is shown in [2] that the degree of perturbation of the matrix A depends substantially on the kind of basis functions $\{W_m(x, t)\}_1^N$ and a suboptimal basis exists in the form of piecewise-constant functions possessing the property that the degree of perturbation of the matrix A is the least. However, the possibility of selection exists even among piecewise-constant functions. For instance, identification results are represented in [2, 3] by using the subspace F_N generated by a system of zeroth-order B-splines (B^0-splines). According to [10], the scalar product operation in the left side of (6) can be interpreted as averaging of the function $D_\nu \tilde{u}(x, t)$ by means of the weight function $W_m(x, t)$. The averaging domain is determined by the domain in which the functions $W_m(x, t)$ do not equal zero. Moreover, the greater the averaging domain, the more, according to [10], will the random component of the quantity $D_\nu \tilde{u}(x, t)$ be suppressed and the degree of perturbation of the elements of the matrix A diminished thereby. Since the B^0-splines differ from zero only in certain subdomains of the domain $\Delta \times \Lambda$, it should be expected that better averaging will be achieved with piecewise-constant functions averaged over the whole domain $\Delta \times \Lambda$. Consequently, it is proposed to select the system of Walsh functions [11] as the piecewise-constant functions.

Represented below are results of investigating the proposed algorithm on the basis of statistical modeling.

RESULTS OF NUMERICAL MODELING

A model of a system with distributed parameters

$$\Theta_1 \frac{\partial^2 u(x, t)}{\partial t^2} + \Theta_2 \frac{\partial^2 u(x, t)}{\partial x^2} + \Theta_3 \frac{\partial u(x, t)}{\partial t} + \Theta_4 \left[\frac{\partial u(x, t)}{\partial x} \right]^2 = f(x, t) \tag{8}$$

was investigated. The true values of the parameters are $\Theta_1^* = 3$, $\Theta_2^* = 2$, $\Theta_3^* = 1$, $\Theta_4^* = 0.5$. It is assumed that the exact solution of the model is $u^*(x, t) = t^2 \ln(1 + x) + x \sin t$, to which the vector of the true parameters Θ^* and the input function $f(x, t) = 2 \ln(1 + x) - x \sin t - t^2/(1 + x)^2 + t^4/(1 + x)^2 + \sin^2 t + 2t^2 \sin t/(1 + x) + 2t \ln(1 + x) + x \cos t$ with appropriate initial and boundary conditions correspond. The measureable quantities u_{mk}' from (2) were modeled by using a weakly correlated sequence ξ_{mk} with standard deviation σ_ξ obtained from a random number generator. The number of measurement points is $M = 40$, $K = 40$. The function $\tilde{u}(x, t)$ was obtained by piecewise-linear interpolation of the values of $\{u_{mk}'\}$.

TABLE 1. Dependence of the Accuracy of the Estimates ε_Θ on the Relative Standard Deviation σ_ξ^0 for Four Jointly Estimatable Parameters $\Theta_1^* = 3$, $\Theta_2^* = 2$, $\Theta_3^* = 1$, and $\Theta_4^* = 0.5$

σ_ξ^0, %	1	3	5	10	15
ε_Θ, %	1,4	4,2	6,9	13,29	19,5

The accuracy of the estimates of the parameters was characterized by the magnitude of the relative error ε_Θ and the level of noisiness of the initial data by the relative standard deviation σ_ξ^0, where

$$\varepsilon_\Theta = \frac{\|\hat{\Theta} - \Theta^*\|_N}{\|\Theta^*\|_N} 100\%, \quad \sigma_\xi^0 = \frac{\sigma_\xi}{\|u^*(x,\ t)\|}. \tag{9}$$

Here

$$\|u^*(x,\ t)\| = \Big[\int_\Delta \int_\Lambda u^{*2}(x,\ t)\,dxdt\Big]^{1/2}, \quad \|\Theta\|_N = \Big(\sum_{n=1}^N \Theta_n^2\Big)^{1/2}.$$

The results of the modeling which are the means of five independent experiments performed (Table 1) show that application of the Walsh functions permits simultaneous estimation of up to four unknown parameters. Upon estimating a large number of parameters the estimates are obtained unstable relative to small deviations from the initial data. In other words, the identification problem becomes incorrect. This is also confirmed in [12]. Moreover, by applying the system of B^0-splines instead of the Walsh functions, the selection is possible of a more suitable system of basis functions, namely, the Walsh functions that permit simultaneous estimation of more parameters for "noisy" data, i.e., an increase in the resolution and thereby the interference-immunity and efficiency of the explicit projection identification algorithm.

NOTATION

$L_2(\Delta \times \Lambda)$, Hilbert space of square-summable functions defined in the space and time intervals Δ and Λ, respectively; $x \in \Delta$, space variable; $t \in \Delta$, time variable; $u^*(x, t)$, $u(x, t)$, exact and noisy output functions of the system; $f(x, t)$, generalized input function; $\{D_\nu\}$, set of partial differential operators; Θ, vector of the constant parameters; F_N, a finite-dimensional subspace of the Hilbert space $L_2(\Delta \times \Lambda)$; $\varphi_n[\cdot]$, given mutually one-to-one nonlinear functions; $\ell(x, t, \Theta)$, residual of the equation; $\langle\cdot, \cdot\rangle$, scalar product in the space $L_2(\Delta \times \Lambda)$; P_N, orthogonal projection operator in the subspace F_N; $\{W_m(x, t)\}_1^N$, a system of basis functions generating F_N; $\{\xi_{mk}\}$, a sequence of random variables simulating the measurement error at points of space $\{x_m\}$ and time $\{t_k\}$; ε_Θ, relative error estimate of the estimates $\hat{\Theta}$ of the vector Θ.

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